Math 210C Lecture 8 Notes

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1 Fractional Ideals in Dedekind Domains

1.1 Fractional ideals

Recall from last time:

Definition 1.1. A **Dedekind domain** is a noetherian domain which is integrally closed and of Krull dimension ≤ 1.

Example 1.1. Dedekind domains of Krull dimension 0 are fields.

Proposition 1.1. *Let A be a Dedekind domain, and let $S \subseteq A$ be multiplicative. Then $S^{-1}A$ is a Dedekind domain.*

Proof. Check the 3 parts of the definition. $\square$

Example 1.2. Dedekind domains may not be UFDs. $\mathbb{Z}[\sqrt{-5}]$ is such an example.

However, there is unique factorization of ideals in Dedekind domains.

Lemma 1.1. *Let A be a noetherian domain, and let $\mathfrak{a} \subseteq A$ be a noetherian domain. There exist nonzero primes $\mathfrak{p}_1, \dots, \mathfrak{p}_k \subseteq A$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{a}$.*

Proof. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be a primary decomposition. Then, setting $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$, we get $\mathfrak{p}_i^{k_i} \subseteq \mathfrak{q}_i$ for some k_i, as A is noetherian. Then $\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r} \subseteq \mathfrak{q}_1, \dots, \mathfrak{q}_r \subseteq \mathfrak{a}$. $\square$

Lemma 1.2. *Let $\mathfrak{p}_i, \mathfrak{p}$ be prime ideals. If $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some i.*

Definition 1.2. A **fractional ideal** in a domain A is a nonzero A-submodule $\mathfrak{a}$ of $Q(A)$ such that there exists nonzero $d \in A$ with $d\mathfrak{a} \subseteq A$.

Example 1.3. Let $a \in Q(A)^\times$. Consider $Aa = (a) \subseteq Q(A)$. Then $a = b/c$, where $b, c \in A$ and $c \neq 0$. Then $c(a) = (ac) = (b) \subseteq A$. Here, (a) is called a **principal fractional ideal**.

Example 1.4. Let $\mathfrak{a} \subseteq A$ be a nonzero ideal. Then $\mathfrak{a}$ is a fractional ideal contained in A.

Lemma 1.3. *Let A be a noetherian domain, and let $\mathfrak{a}$ be a nonzero A-submodule of $Q(A)$. Then $\mathfrak{a}$ is a fractional ideal if and only if $\mathfrak{a}$ is finitely generated as an A-module.*

Proof. Let $\mathfrak{a}$ be finitely generated. THen $\mathfrak{a} = A(b-1/c_1 \ldots, b_k/c_k \subseteq Q(A)$. Take $d = c_1, \ldots, c_k \in \mathbb{A} \setminus \{0\}$. Then $d\mathfrak{a} \subseteq A$, so $\mathfrak{a}$ is fractional.

Conversely, suppose $\mathfrak{a}$ is fractional. Then there exists $d \in A \setminus \{0\}$ such that $d\mathfrak{a} \subseteq A$. Since A is noetherian, $d\mathfrak{a} = (b_1, \ldots, b_k$, so $\mathfrak{a} = A(b_1/d, \ldots, b_k/d)$ is finitely generated. □

1.2 Inverses of fractional ideals

Definition 1.3. Let A be a domain with $K = Q(A)$ and fractional ideals $\mathfrak{a}, \mathfrak{b}$. The **inverse** of $\mathfrak{a}$ is $\mathfrak{a}^{-1} = \{b \in Q(A) : b\mathfrak{a} \subseteq A\}$. The **product** $\mathfrak{ab}$ is the A submodule of $Q(A)$ generated by $\{ab : a \in \mathfrak{a}, b \in \mathfrak{b}\}$.

Example 1.5. Let $\mathfrak{a} = (3/4) = (1/4) \subseteq Q(\mathbb{Z})$. Then $\mathfrak{a}^{-1} = (4/3)$.

$I(A)$, the set of fractional ideals of A, is a monoid with identity A under this operation.

Lemma 1.4. *Let A be a domain, and let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of A. Then $\mathfrak{a}^{-1}, \mathfrak{a}+\mathfrak{b}, \mathfrak{ab}$, and $\mathfrak{a} \cap \mathfrak{b}$ are fractional ideals.*

Proof. Let $c, d \in A \setminus \{0\}$ such that $c\mathfrak{a} \subseteq \mathfrak{d}\mathfrak{b} \subseteq$. Then $c(\mathfrak{a} \cap \mathfrak{b}), cd(\mathfrak{a}+\mathfrak{b}), cd\mathfrak{ab} \subseteq A$.

$\mathfrak{a}^{-1} \neq 0$ since there exists $c \in A \setminus \{0\}$ such that $c\mathfrak{a} \subseteq A$. So $c \in \mathfrak{a}^{-1}$. Let $a \in \mathfrak{a}$ with $a \neq 0$. Then $a = e/f$ with $e, f \in A$ and $f \neq 0$. Then $e \in \mathfrak{a} \cap A$. Now $e\mathfrak{a}^{-1} \subseteq a\mathfrak{a}^{-1} \subseteq A$ by definition of $\mathfrak{a}^{-1}$. So $\mathfrak{a}^{-1}$ is a fractional ideal. □

Definition 1.4. A fractional ideal $\mathfrak{a}$ of A is **invertible** if there exists a fractional ideal $\mathfrak{b}$ of A such that $\mathfrak{ab} = A$.

Lemma 1.5. *Let $\mathfrak{a}$ be a fractional ideal of A. Then $\mathfrak{a}$ is invertible iff $\mathfrak{a}\mathfrak{a}^{-1} = A$.*

Proof. ($\implies$): If $\mathfrak{a}$ is invertible, then let $\mathfrak{ab} = A$. Then $\mathfrak{b} \subseteq \mathfrak{a}^{-1}$. Also, $A = \mathfrak{ab} \subseteq \mathfrak{a}\mathfrak{a}^{-1} \subseteq A$. So these inclusions are actually equalities. □

Example 1.6. Let $\mathbb{Q}[x,y] \supseteq (x,y) = \mathfrak{m}$. What is $\mathfrak{m}^{-1}$? Note that $f \in \mathbb{Q}(x,y)^\times$ satisfies $f\mathfrak{m} \subseteq \mathbb{Q}[x,y]$ if and only if $fx \in \mathbb{Q}[x,y]$ and $fy \in \mathbb{Q}[x,y]$. This is if and only if $f \in \mathbb{Q}[x,y]$. So $\mathfrak{m}^{-1} = \mathbb{Q}[x,y]$, and $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$. This is a bit weird; this is because $\mathbb{Q}[x,y]$ is not a Dedekind domain (Krull dimension 2).

Remark 1.1. If $a \in Q(A)^\times$, then $(a)^{-1} = (a^{-1})$, and $(a) \cdot (a^{-1}) = A$.

Lemma 1.6. *Every fractional ideal in a PID is principal.*

Proof. If $\mathfrak{a}$ is fractional, then $d\mathfrak{a} \subseteq A$ is an ideal, generated by b. Then $\mathfrak{a} = (b/d)$. □

Lemma 1.7. *If A is a Dedekind domain, and $\mathfrak{p} \subseteq A$ is a nonzero prime ideal , then $\mathfrak{p}\mathfrak{p}^{-1} = A$.*

Proof. Let $a \in \mathfrak{p}$ with $a \neq 0$. There exists a minimal $k \geq 1$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq (a)$ (where $\mathfrak{p}_i$ are prime and $\neq \mathfrak{a}$) by the lemma. Since $\mathfrak{p}_1 \cdots \mathfrak{p}_j \subseteq \mathfrak{p}$, $\mathfrak{p} = \mathfrak{p}_k$ without loss of generality. By the minimality of k, there exists a b in $\mathfrak{p}_1 \cdots \mathfrak{p}_{k-1}$ such that $b \notin (a)$. Then $a^{-1}b \notin A$. Then $a^{-1}b\mathfrak{p} \subseteq a^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq A$. So $a^{-1}b \in \mathfrak{p}^{-1}$.

Note that $A \supseteq \mathfrak{p}^{-1}\mathfrak{p} \supseteq \mathfrak{p}$. If these are equal, then $a^{-1}b\mathfrak{p} \subseteq \mathfrak{p}$, and $\mathfrak{p}$ is finitely generated. So $a^{-1}b$ is integral over A. Since A is integrally closed, $a^{-1}b \in A$. But $a^{-1}b \notin A$, which is a contradiction. So $\mathfrak{p}^{-1}\mathfrak{p} \supsetneq \mathfrak{p}$, where $\mathfrak{p}$ is maximal (because A has Krull dimension 1), so $\mathfrak{p}^{-1}\mathfrak{p} = A$. $\square$